

An Uniqueness Theorem on the Eigenvalues of Spherically Symmetric Interior Transmission Problem in Absorbing Medium

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Abstract

We study the asymptotic distribution of the eigenvalues of interior transmission problem in absorbing medium. We apply Cartwright's theory and the technique from entire function theory. We find a Weyl's type of density theorem on counting the eigenvalues for spherically symmetric interior transmission problem in absorbing medium. Given a sufficient quantity of transmission eigenvalues, we obtain limited uniqueness on the refraction index as an uniqueness problem in entire function theory.

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1 Preliminaries

In this short note, we study the eigenvalues of the interior transmission problem with a twice differentiable absorbing refraction index

$$n_1(x) := \epsilon_1(x) + i \frac{\gamma_1(x)}{k} : \quad (1.1)$$

$$\begin{cases} \Delta w + k^2(\epsilon_1(x) + i \frac{\gamma_1(x)}{k})w = 0, & \text{in } B; \\ \Delta v + k^2(\epsilon_0 + i \frac{\gamma_0}{k})v = 0 & \text{in } B; \\ w = v, & \text{on } \partial B; \\ \frac{\partial w}{\partial r} = \frac{\partial v}{\partial r} & \text{on } \partial B, \end{cases} \quad (1.2)$$

where $B := \{|x| \leq 1, x \in \mathbb{R}^3\}$, $w, v \in L^2(B)$, $w - v \in H_0^2(B)$, $k \in \mathbb{C}$. We consider the spherical perturbations for (1.2) by setting $\epsilon_1(x) = \epsilon_1(r) > 0$ and $\gamma_1(x) = \gamma_1(r) > 0$, $\forall r \in [0, 1]$; ϵ_0 and γ_0 are positive constants and $n_1(r) = \epsilon_0 + i \frac{\gamma_0}{k}$, $r \geq 1$.

The interior transmission eigenvalues play a role in the inverse scattering theory both in numerical computation and in theoretical purpose. See Colton and Kress [6] and Colton, Päiväranta and Sylvester [5] for the historic and theoretical context. Moreover, the eigenvalues of the interior transmission problem is directly connected to the zeros of scattering amplitude. They are zeros of the integral average of the scattering amplitude. We refer to McLaughlin and Polyakov [10]. In this paper we use the analysis in Levin [8] to discuss the zeros of an asymptotically almost periodic function along the real axis.

Let us consider the solutions of (1.2) with of the following form:

$$v(r) = c_1 j_0(k \tilde{n}_0 r); \quad (1.3)$$

$$w(r) = c_2 \frac{y(r)}{r}, \quad (1.4)$$

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where $\tilde{n}_0 := (\epsilon_0 + i\frac{\gamma_0}{k})^{\frac{1}{2}}$, j_0 is a spherical Bessel function of order zero and $y(r)$ is a solution of

$$y'' + k^2(\epsilon_1(r) + i\frac{\gamma_1(r)}{k})y = 0; \quad (1.5)$$

$$y(0) = 0; y'(0) = 1. \quad (1.6)$$

The existence of the (1.3) and (1.4) is provided by

$$D(k) := \det \begin{pmatrix} y(1) & -j_0(k\tilde{n}_0) \\ \{\frac{y(r)}{r}\}'|_{r=1} & -j_0'(k\tilde{n}_0 r)|_{r=1} \end{pmatrix} = 0. \quad (1.7)$$

The interior transmission eigenvalues are the zeros of such a functional determinant. Furthermore, it is well-known that $D(k)$ is an entire function of exponential type bounded on the real axis. Therefore, it is in Cartwright's class of functions. Such a function has many advantages. We refer to Levin [8, 9] for an introduction.

Let us write

$$\Phi(r; k) = rw(r), \quad \Phi_0(r; k) = rv(r).$$

The interior transmission eigenvalues of (1.2) coincide with the eigenvalues of the nonstandard boundary value problem:

$$\begin{cases} \Phi'' + k^2(\epsilon_1(r) + i\frac{\gamma_1(r)}{k})\Phi = 0, & 0 < r < 1; \\ \Phi(0) = 0, D(k) = 0. \end{cases} \quad (1.8)$$

Moreover, (1.8) is equivalent to the following system.

$$\begin{cases} \Phi'' + k^2(\epsilon_1(r) + i\frac{\gamma_1(r)}{k})\Phi = 0, & 0 < r < 1; \\ \Phi_0'' + k^2(\epsilon_0 + i\frac{\gamma_0}{k})\Phi_0 = 0, & 0 < r < 1; \\ \Phi(0) = \Phi_0(0), \Phi(1) = \Phi_0(1), \Phi'(1) = \Phi_0'(1). \end{cases} \quad (1.9)$$

The zeros of the functional determinant $D(k)$ are then the eigenvalues of (1.8). In particular, if k_j is an eigenvalue of (1.8), then $D(k_j) = 0$; if $D(k_j) = 0$, then $y(r; k_j)$ is an eigenfunction of (1.8) with eigenvalue k_j . In that case, $y(1; k_j) = \Phi_0(1)$ by the third line of (1.9). The formulation (1.8) and (1.9) are founded in [1].

To understand the analytic behavior of the determinant $D(k)$, we study the asymptotic solution of (1.5) and (1.6). In this case, we use the theory provided in Erdelyi [7]. In particular, we have a set of fundamental solutions $y_1(r)$, $y_2(r)$ such that in a sectorial region S

$$y_j(r; k) = Y_j(r)[1 + O(\frac{1}{k})]; \quad (1.10)$$

$$y_j'(r; k) = Y_j'(r)[1 + O(\frac{1}{k})], \quad (1.11)$$

as $|k| \rightarrow \infty$ in S , uniformly for $0 \leq r \leq a$ and for $\arg k$, where

$$Y_j(r) = \exp\{\beta_{0j}k + \beta_{1j}\}, \quad (1.12)$$

where β_{0j} , β_{1j} satisfy

$$(\beta_{0j}')^2 + \epsilon_1(r) = 0; \quad (1.13)$$

$$2\beta_{0j}'\beta_{1j}' + i\gamma_1 + \beta_{0j}'' = 0. \quad (1.14)$$

$$\beta_{0j}(r) = \pm i \int_0^r \sqrt{\epsilon_1(\rho)} d\rho + E; \quad (1.15)$$

$$\beta_{1j}(r) = \mp \frac{1}{2} \int_0^r \frac{\gamma_1}{\sqrt{\epsilon_1(\rho)}} d\rho + \log[\epsilon_1(r)]^{\frac{-1}{4}} + F, \quad (1.16)$$

where E , F are constants. The sectorial region $S \subset \mathbb{C}$ is characterized by the condition

$$\Re\{ki(\epsilon_1(r))^{\frac{1}{2}}\} \neq 0. \quad (1.17)$$

That is

$$S = \{k \in \mathbb{C} \mid \Im k \neq 0\}. \quad (1.18)$$

Therefore, any solution to (1.5) is of the form

$$y(r; k) = \alpha Y_1(r)[1 + O(\frac{1}{k})] + \beta Y_2(r)[1 + O(\frac{1}{k})]. \quad (1.19)$$

We use the initial condition (1.6) to obtain

$$\begin{aligned} y(r; k) &= \frac{1}{2ik[\epsilon_1(0)\epsilon_1(r)]^{\frac{1}{4}}} \exp\{ik \int_0^r \sqrt{\epsilon_1(\rho)} d\rho - \frac{1}{2} \int_0^r \frac{\gamma_1(\rho)}{\sqrt{\epsilon_1(\rho)}} d\rho\} [1 + O(\frac{1}{k})] \\ &\quad - \frac{1}{2ik[\epsilon_1(0)\epsilon_1(r)]^{\frac{1}{4}}} \exp\{-ik \int_0^r \sqrt{\epsilon_1(\rho)} d\rho + \frac{1}{2} \int_0^r \frac{\gamma_1(\rho)}{\sqrt{\epsilon_1(\rho)}} d\rho\} [1 + O(\frac{1}{k})], \end{aligned} \quad (1.20)$$

when $|k| \rightarrow \infty$ in S . Similarly, we use (1.11) to obtain the asymptotics

$$\begin{aligned} y'(r; k) &= \frac{1}{2} [\frac{\epsilon_1(r)}{\epsilon_1(0)}]^{\frac{1}{4}} \exp\{ik \int_0^r \sqrt{\epsilon_1(\rho)} d\rho - \frac{1}{2} \int_0^r \frac{\gamma_1(\rho)}{\sqrt{\epsilon_1(\rho)}} d\rho\} [1 + O(\frac{1}{k})] \\ &\quad + \frac{1}{2} [\frac{\epsilon_1(r)}{\epsilon_1(0)}]^{\frac{1}{4}} \exp\{-ik \int_0^r \sqrt{\epsilon_1(\rho)} d\rho + \frac{1}{2} \int_0^r \frac{\gamma_1(\rho)}{\sqrt{\epsilon_1(\rho)}} d\rho\} [1 + O(\frac{1}{k})], \end{aligned} \quad (1.21)$$

when $|k| \rightarrow \infty$ in S .

Let us set

$$A := \sqrt{\epsilon_0}, \quad B := \int_0^1 \sqrt{\epsilon_1(\rho)} d\rho, \quad C := \frac{1}{2} \frac{\gamma_0}{\sqrt{\epsilon_0}}, \quad D := \frac{1}{2} \int_0^1 \frac{\gamma_1(\rho)}{\sqrt{\epsilon_1(\rho)}} d\rho. \quad (1.22)$$

We state the main result of this paper.

Theorem 1.1 *Let*

$$\Lambda_1 := \{z \in \mathbb{C} \mid |\arg(z)| < \epsilon\}; \quad (1.23)$$

$$\Lambda_2 := \{z \in \mathbb{C} \mid |\arg(z) - \pi| < \epsilon\}, \quad \forall \epsilon > 0. \quad (1.24)$$

Let n_1^j , $j = 1, 2$, be two refraction indices and $D^j(k)$ be the determinant corresponding to n_1^j . If the zeros of $D^1(k)$ and $D^2(k)$ coincide in either Λ_1 or Λ_2 , then $\epsilon_1^1(r) \equiv \epsilon_1^2(r)$.

We use the vocabulary from entire function to describe the distribution of the zeros of the functional determinant $D(k)$. We refer such a theory to Levin [8, 9].

Definition 1.2 *Let $f(z)$ be an entire function of order ρ . We use $N(f, \alpha, \beta, r)$ to denote the number of the zeros of $f(z)$ inside the angle $[\alpha, \beta]$ and $|z| \leq r$; we define the density function*

$$\Delta_f(\alpha, \beta) := \lim_{r \rightarrow \infty} \frac{N(f, \alpha, \beta, r)}{r^\rho}, \quad (1.25)$$

and

$$\Delta_f(\beta) := \Delta_f(\alpha_0, \beta), \quad (1.26)$$

with some fixed $\alpha_0 \notin E$ such that E is at most a countable set.

Theorem 1.3 *The determinant $D(k)$ is an entire function of order 1 and of type $A + B$. In particular,*

$$\Delta_D(-\epsilon, \epsilon) = \Delta_D(\pi - \epsilon, \pi + \epsilon) = \frac{A + B}{\pi}. \quad (1.27)$$

2 The Proofs

We need a few lemmas.

Lemma 2.1 *There exists a constant M and $k_0 > 0$ such that*

$$\left| \frac{y(1; k)}{y'(1; k)} \right| < M; \left| \frac{j_0(k\tilde{n}_0)}{\partial_r j_0(k\tilde{n}_0 r)|_{r=1}} \right| < M, \forall k \in 0 + i\mathbb{R}^+, |k| > k_0. \quad (2.1)$$

Proof We start with (1.20). We compute the following quantity from (1.10), (1.11) and (1.20).

$$\frac{y(1; k)}{y'(1; k)} = \frac{e^{ikB-D}[1 + O(\frac{1}{k})] - e^{-ikB+D}[1 + O(\frac{1}{k})]}{e^{ikB-D}[ik\epsilon_0 - \frac{\gamma_0}{2\sqrt{\epsilon_0}}][1 + O(\frac{1}{k})] + e^{-ikB+D}[ik\epsilon_0 - \frac{\gamma_0}{2\sqrt{\epsilon_0}}][1 + O(\frac{1}{k})]}. \quad (2.2)$$

Let $k = iy \in 0 + i\mathbb{R}^+$. Hence,

$$\frac{y(1; iy)}{y'(1; iy)} = \frac{e^{-yB-D}[1 + O(\frac{1}{iy})] - e^{yB+D}[1 + O(\frac{1}{iy})]}{e^{-yB-D}[-y\epsilon_0 - \frac{\gamma_0}{2\sqrt{\epsilon_0}}][1 + O(\frac{1}{iy})] + e^{yB+D}[-y\epsilon_0 - \frac{\gamma_0}{2\sqrt{\epsilon_0}}][1 + O(\frac{1}{iy})]}, \quad (2.3)$$

which is bounded for large $|y|$. Hence, the first statement is proved. The second one can be proved similarly. \square

Definition 2.2 *Let $f(z)$ be an integral function of finite order ρ in the angle $[\theta_1, \theta_2]$. We call the following quantity as the indicator of the function $f(z)$.*

$$h_f(\theta) := \lim_{r \rightarrow \infty} \frac{\ln |f(re^{i\theta})|}{r^\rho}, \theta_1 \leq \theta \leq \theta_2. \quad (2.4)$$

Lemma 2.3 *Let f, g be two entire functions. Then, the following two inequalities hold.*

$$h_{fg}(\theta) = h_f(\theta) + h_g(\theta), \text{ if one limit exists;} \quad (2.5)$$

$$h_{f+g}(\theta) \leq \max_{\theta} \{h_f(\theta), h_g(\theta)\}, \quad (2.6)$$

where if the indicator of the two summands are not equal at some θ_0 , then the equality holds in (2.6).

Proof We can find these in [8, p.51]. \square

Definition 2.4 *The following quantity is called the width of the indicator diagram of entire function f :*

$$d = h_f\left(\frac{\pi}{2}\right) + h_f\left(-\frac{\pi}{2}\right). \quad (2.7)$$

The distribution on the zeros of an entire function is described precisely by the following Cartwright's theorem [3, 4, 8, 9]. The following statements are from Levin [8, ch.5, sec.4].

Theorem 2.5 (Cartwright) *If an entire function of exponential type satisfies one of the following conditions:*

$$\text{the integral } \int_0^\infty \frac{\ln |f(x)f(-x)|}{1+x^2} dx \text{ exists, and } h_f(0) = h_f(\pi) = 0, \quad (2.8)$$

$$\text{the integral } \int_{-\infty}^\infty \frac{\ln |f(x)|}{1+x^2} dx < \infty. \quad (2.9)$$

$$\text{the integral } \int_{-\infty}^\infty \frac{\ln^+ |f(x)|}{1+x^2} dx \text{ exists.} \quad (2.10)$$

$$|f(x)| \text{ is bounded on the real axis.} \quad (2.11)$$

$$|f(x)| \in \mathcal{L}^p(-\infty, \infty), \quad (2.12)$$

then

1. $f(z)$ is of class A and is of completely regular growth and its indicator diagram is an interval on the imaginary axis.
2. all of the zeros of the function $f(z)$, except possibly those of a set of zero density, lie inside arbitrarily small angles $|\arg z| < \epsilon$ and $|\arg z - \pi| < \epsilon$, where the density

$$\Delta_f(-\epsilon, \epsilon) = \Delta_f(\pi - \epsilon, \pi + \epsilon) = \lim_{r \rightarrow \infty} \frac{N(f, -\epsilon, \epsilon, r)}{r} = \lim_{r \rightarrow \infty} \frac{N(f, \pi - \epsilon, \pi + \epsilon, r)}{r}, \quad (2.13)$$

is equal to $\frac{d}{2\pi}$, where d is the width of the indicator diagram in (2.7). Furthermore, the limit $\delta = \lim_{r \rightarrow \infty} \delta(r)$ exists, where

$$\delta(r) := \sum_{\{|a_k| < r\}} \frac{1}{a_k}; \quad (2.14)$$

3. moreover,

$$\Delta_f(\epsilon, \pi - \epsilon) = \Delta_f(\pi + \epsilon, -\epsilon) = 0, \quad (2.15)$$

4. the function $f(z)$ can be represented in the form

$$f(z) = cz^m e^{iCz} \lim_{r \rightarrow \infty} \prod_{\{|a_k| < r\}} \left(1 - \frac{z}{a_k}\right), \quad (2.16)$$

where c, m, B are constants and C is real.

5. the indicator function of f is of the form

$$h_f(\theta) = \sigma |\sin \theta|. \quad (2.17)$$

The last statement is found at Levin [9, p.126]. We use these to compute the indicator function of $D(k)$.

Proposition 2.6

$$h_D(\theta) = (A + B) |\sin \theta|, \quad \theta \in [0, 2\pi]. \quad (2.18)$$

Proof As we see from (1.7),

$$D(k) = y'(1, k) \partial_r j_0(k \tilde{n}_0 r)|_{r=1} \left\{ \frac{j_0(k \tilde{n}_0)}{\partial_r j_0(k \tilde{n}_0 r)|_{r=1}} - \frac{y(1; k)}{y'(1; k)} \left[1 + \frac{j_0(k \tilde{n}_0)}{\partial_r j_0(k \tilde{n}_0 r)|_{r=1}} \right] \right\}. \quad (2.19)$$

For $k = iy$ and $|y| > k_0$, we have that

$$D(iy) = y'(1; iy) \partial_r j_0(iy \tilde{n}_0 r)|_{r=1} \left\{ \frac{j_0(iy \tilde{n}_0)}{\partial_r j_0(iy \tilde{n}_0 r)|_{r=1}} - \frac{y(1; iy)}{y'(1; iy)} \left[1 + \frac{j_0(iy \tilde{n}_0)}{\partial_r j_0(iy \tilde{n}_0 r)|_{r=1}} \right] \right\}, \quad (2.20)$$

where the items inside the bracket are bounded by Lemma 2.1. Furthermore, we compute

$$\begin{aligned} h_{y'(1; k)}(\pm \frac{\pi}{2}) &= \lim_{y \rightarrow \infty} \frac{\ln |y'(1; iy)|}{|y|} \\ &= \lim_{y \rightarrow \infty} \frac{\ln |e^{-yB-D} [-y\epsilon_0 - \frac{\gamma_0}{2\sqrt{\epsilon_0}}] [1 + O(\frac{1}{iy})] + e^{yB+D} [-y\epsilon_0 - \frac{\gamma_0}{2\sqrt{\epsilon_0}}] [1 + O(\frac{1}{iy})]|}{|y|} \\ &= B. \end{aligned} \quad (2.21)$$

Similarly,

$$h_{\partial_r j_0(k \tilde{n}_0 r)|_{r=1}}(\pm \frac{\pi}{2}) = A. \quad (2.22)$$

Now we use (2.5) to obtain

$$h_D(\pm \frac{\pi}{2}) = (A + B). \quad (2.23)$$

Hence, (2.17) says

$$h_D(\theta) = (A + B) |\sin \theta|. \quad (2.24)$$

□

Proof of Theorem 1.3 The indicator diagram of $D(k)$ has width $2(A+B)$. (2.13) and (2.23) suggests that

$$\Delta_D(-\epsilon, \epsilon) = \Delta_D(\pi - \epsilon, \pi + \epsilon) = \frac{A+B}{\pi}. \quad (2.25)$$

□

So we have this quantity of zeros in Λ_1 and Λ_2 .

3 Proof of Theorem 1.1

Let k_j be a common interior transmission eigenvalue of refraction index $n_1^1(r)$ and $n_1^2(r)$. Let $D^i(k)$ be the corresponding functional determinant of the index $n_1^i(r)$; $y^i(r; k)$ be the solution. From Theorem 1.3 and the assumption of Theorem 1.1, we have

$$B^1 = B^2, \quad (3.1)$$

where

$$B^i := \int_0^1 \sqrt{\epsilon_1^i(\rho)} d\rho.$$

Moreover, we use the boundary condition in (1.9) to see

$$y^1(1; k_j) = y^2(1; k_j), \quad \forall k_j. \quad (3.2)$$

Let

$$F(k) := y^1(1; k) - y^2(1; k), \quad (3.3)$$

which is an entire function. From (1.20) and (2.4), we see $y^i(1; k)$ is an entire function of exponential type B^i and

$$h_{y^i}(\theta) = B^i |\sin \theta|. \quad (3.4)$$

Here, we see that $h_{y^i}(\theta)$ is a continuous function of θ . Therefore, applying (2.6) and (3.1),

$$h_F(\theta) \leq B^1 |\sin \theta|. \quad (3.5)$$

Hence, the indicator diagram of $F(k)$ is equal to

$$h_F\left(\frac{\pi}{2}\right) + h_F\left(-\frac{\pi}{2}\right) \leq 2B^1. \quad (3.6)$$

However, this is not possible due to the following uniqueness theorem for the entire function of the exponential type. This is the Carlson's theorem from Levin [8, p.190].

Theorem 3.1 *Let $F(z)$ be holomorphic and at most of normal type with respect to the proximate order $\rho(r)$ in the angle $\alpha \leq \arg z \leq \alpha + \pi/\rho$ and vanish on a set $N := \{a_k\}$ in this angle, with angular density $\Delta_N(\psi)$. Let*

$$H_N(\theta) := \pi \int_{\alpha}^{\alpha + \pi/\rho} \sin |\psi - \theta| d\Delta_N(\psi), \quad (3.7)$$

when ρ is integral. Then, if $F(z)$ is not identically zero,

$$h_F(\alpha) + h_F(\alpha + \pi/\rho) \geq H_N(\alpha) + H_N(\alpha + \pi/\rho). \quad (3.8)$$

In this paper, we consider $\rho \equiv 1$, $\alpha = -\frac{\pi}{2}, \frac{\pi}{2}$. Let N here be the common zeros either in Λ_1 or in Λ_2 . From (2.13) and (2.15), we have

$$H_N\left(-\frac{\pi}{2}\right) + H_N\left(\frac{\pi}{2}\right) = 2(A + B^1). \quad (3.9)$$

Therefore, we conclude from (3.6), (3.16) and Theorem 3.4 that $F(k) \equiv 0$ and

$$y^1(1; k) \equiv y^2(1; k). \quad (3.10)$$

Similarly, we repeat the argument from (3.2) to (3.16), we can show

$$(y^1)'(1; k) \equiv (y^2)'(1; k). \quad (3.11)$$

The zeros of $y^i(1; k)$ are the eigenvalues of the equation

$$\begin{cases} (y^i)'' + k^2 n_1^i(r) y^i = 0, & 0 \leq r \leq 1; \\ y^i(0) = 0, & y^i(1; k) = 0, \end{cases} \quad (3.12)$$

while the zeros of $(y^i)'(1; k)$ are the eigenvalues of the equation

$$\begin{cases} (y^i)'' + k^2 n_1^i(r) y^i = 0, & 0 \leq r \leq 1; \\ y^i(0) = 0, & (y^i)'(1; k) = 0. \end{cases} \quad (3.13)$$

Let us consider the perturbation to these two sets of zeros.

Definition 3.2 *Let us define*

$$\mathcal{Y}(r; k) := y^1(r; k) - y^2(r; k). \quad (3.14)$$

The following lemma holds.

Lemma 3.3 *Let k be a common interior transmission eigenvalue for index n_1^1 and n_1^2 . Then, $\mathcal{Y}(1; k)$ satisfies the following boundary value problem.*

$$\mathcal{Y}''(r; k) + (k^2 \epsilon_1^1 + ik\gamma_1^1) \mathcal{Y}(r; k) + (k^2 \epsilon_d + ik\gamma_d) y^2(r; k) = 0, \quad 0 \leq r \leq 1; \quad (3.15)$$

$$(y^2)''(r; k) + (k^2 \epsilon_1^2 + ik\gamma_1^2) y^2(r; k) = 0, \quad 0 \leq r \leq 1; \quad (3.16)$$

$$\mathcal{Y}(1; k) = 0; \quad (3.17)$$

$$\mathcal{Y}'(1; k) = 0, \text{ where } \epsilon_d := \epsilon_1^1 - \epsilon_1^2, \gamma_d := \gamma_1^1 - \epsilon_1^2. \quad (3.18)$$

The solutions of

$$\begin{cases} y^1(1; k) = y^2(1; k); \\ (y^1)'(1; k) = (y^2)'(1; k). \end{cases} \quad (3.19)$$

are among the interior transmission eigenvalues of problem (3.15)- (3.18).

Proof We note that (3.17) and (3.18) are satisfied as in (3.2); (3.15) and (3.16) are justified by (1.5). \square

The system (3.15)- (3.18) has a perturbation theory to merely finitely many eigenvalues. Such a theory is established in [2, sec.2]. Let us review as follows.

Let us define

$$u := w - v. \quad (3.20)$$

We rewrite (1.2) as

$$\Delta u + (k^2 \epsilon_1 + ik\gamma_1) u + (k^2 \epsilon_c + ik\gamma_c) v = 0; \quad (3.21)$$

$$\Delta v + (k^2 \epsilon_0 + ik\gamma_0) v = 0; \quad (3.22)$$

$$u = 0, \text{ on } \partial B; \quad (3.23)$$

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial B, \text{ where } \epsilon_c := \epsilon_1 - \epsilon_0; \gamma_c := \gamma_1 - \epsilon_0. \quad (3.24)$$

The equation makes sense for $u \in H_0^2(B)$ and $v \in L^2(B)$ such that $\Delta v \in L^2(B)$.

Setting

$$X(B) := H_0^2(B) \times \{v \in L^2(B) \mid \Delta v \in L^2(B)\},$$

we can define the linear operators \mathbb{A} , \mathbb{B}_γ and

$$\mathbb{D}_\epsilon : L^2(B) \times L^2(B) \mapsto L^2(B) \times L^2(B)$$

by

$$\mathbb{A} := \begin{pmatrix} \Delta_{00} & 0 \\ 0 & \Delta \end{pmatrix}; \quad (3.25)$$

$$\mathbb{B}_\gamma := \begin{pmatrix} i\gamma_1 & i\gamma_c \\ 0 & i\gamma_0 \end{pmatrix}; \quad (3.26)$$

$$\mathbb{D}_\epsilon := \begin{pmatrix} \epsilon_1 & \epsilon_c \\ 0 & \epsilon_0 \end{pmatrix}, \quad (3.27)$$

where Δ_{00} is the Laplacian acting on function in $H_0^2(B)$, i.e. with the zero Cauchy data on ∂B . Let

$$p := \begin{pmatrix} u \\ v \end{pmatrix}$$

and note that the domain of \mathbb{A} is $X(B)$ and \mathbb{A} is an unbounded densely defined operator in $L^2(B) \times L^2(B)$.

Using the results in [2], we can easily show that

$$D_\epsilon^{-1} = \frac{1}{\epsilon_0 \epsilon_1} \begin{pmatrix} \epsilon_0 & -\epsilon_c \\ 0 & \epsilon_1 \end{pmatrix} \quad (3.28)$$

and that the transmission eigenvalues of (1.2) is the quadratic eigenvalues of the equation

$$\mathbb{A}p + k\mathbb{B}_\gamma p + k^2\mathbb{D}_\epsilon p = 0, p = \begin{pmatrix} u \\ v \end{pmatrix} \in L^2(B) \times L^2(B). \quad (3.29)$$

According to [2], the eigenvalue problem (1.2) is equivalent to the eigenvalue problem for the closed unbounded operator

$$\mathbb{T}_{\epsilon,\gamma} := \mathbb{I}_{\epsilon,\gamma}^{-1} \mathbb{K}, \quad (3.30)$$

where,

$$\mathbb{K} := \begin{pmatrix} \mathbb{A} & 0 \\ 0 & \mathbb{I} \end{pmatrix} \quad (3.31)$$

and

$$\mathbb{I}_{\epsilon,\gamma} := \begin{pmatrix} -\mathbb{B}_\gamma & -\mathbb{I} \\ \mathbb{D}_\epsilon & 0 \end{pmatrix}, \quad (3.32)$$

where $\mathbb{I} : L^2(B) \times L^2(B) \mapsto L^2(B) \times L^2(B)$ is the identity operator. Moreover,

$$\mathbb{I}_{\epsilon,\gamma}^{-1} = \mathbb{D}_\epsilon^{-1} \begin{pmatrix} 0 & -\mathbb{I} \\ -\mathbb{D}_\epsilon & -\mathbb{B}_\gamma \end{pmatrix}, \quad (3.33)$$

which is bounded in $L^2(B) \times L^2(B)$. Therefore, $\mathbb{T}_{\epsilon,\gamma}$ is closed in $[L^2(B) \times L^2(B)]^2$.

Now we define the operator

$$\mathbb{P} := \mathbb{T}_{\epsilon,\gamma} - \mathbb{T}_{\epsilon,\gamma=0}. \quad (3.34)$$

Consequently,

$$\mathbb{P} = \begin{pmatrix} 0 & 0 \\ 0 & -\mathbb{D}_\epsilon^{-1} \mathbb{B}_\gamma \end{pmatrix}. \quad (3.35)$$

There is a stability for finitely many interior transmission eigenvalues whenever the generalized norm

$$\hat{\delta}(\mathbb{T}_{\epsilon,\gamma}, \mathbb{T}_{\epsilon,\gamma=0}) \quad (3.36)$$

is controlled. We apply the (24) in [2].

$$\hat{\delta}(\mathbb{T}_{\epsilon,\gamma}, \mathbb{T}_{\epsilon,\gamma=0}) \leq \|\mathbb{P}\| \leq \|\mathbb{D}_\epsilon^{-1} \mathbb{B}_\gamma\|. \quad (3.37)$$

In [2], they perturbed the non-absorbing media, $(\gamma_1 = 0, \gamma_0 = 0)$, to conclude the existence of the finitely many interior transmission eigenvalues in absorbing media, in which (γ_1, γ_0) is small. In particular, they conclude as follows.

Theorem 3.4 (Cakoni, Colton, Haddar) *Let $\epsilon_0 \in L^\infty([0, 1])$ and satisfy $\epsilon_0(r) \geq \theta_0 > 0$, $\epsilon_1(r) \geq \theta_1 > 0$ and $\epsilon_c := \epsilon_1 - \epsilon_0 > 0$ almost every in $[0, 1]$. Let k_j , $j = 0, \dots, l$ be the first $l + 1$ real positive transmission eigenvalues, counted according to its multiplicity, corresponding to (1.2) in non-absorbing media. Then, for any $\sigma > 0$, there exists a*

$$\eta := \frac{\sup_{[0,1]} \epsilon_0 + \sup_{[0,1]} \epsilon_1}{4 \inf_{[0,1]} \epsilon_0 \inf_{[0,1]} \epsilon_1} \left\{ \sup_{[0,1]} \gamma_0 + \sup_{[0,1]} \gamma_1 \right\} > 0, \quad (3.38)$$

depending on σ , such that there exists $l + 1$ transmission eigenvalues of (1.2) in absorbing media, $\epsilon_1 > 0$, $\gamma_1 > 0$, in the σ -neighborhood of k_j , $j = 0, \dots, l$, whenever η is small enough.

For the application in this paper, we consider the perturbation operator

$$\mathbb{P}' := \mathbb{T}_{\epsilon, \gamma} - \mathbb{T}_{\epsilon, \gamma'}. \quad (3.39)$$

We note that the existence of the transmission interior eigenvalues to (1.2) and its distributional rule is already described by Cartwright's theory as in Theorem 1.3. Hence, we consider the perturbation from one absorbing index $n_1 = \epsilon_1 + i\frac{\gamma_1}{k}$ to the other one, with ϵ_1 fixed, $n'_1 = \epsilon_1 + i\frac{\gamma'_1}{k}$. Now we compute

$$\|\mathbb{P}'\| = \|\mathbb{T}_{\epsilon, \gamma} - \mathbb{T}_{\epsilon, \gamma'}\| \leq \|\mathbb{D}_\epsilon^{-1}(\mathbb{B}_{\gamma'} - \mathbb{B}_\gamma)\|. \quad (3.40)$$

Moreover,

$$\|\mathbb{D}_\epsilon^{-1}(\mathbb{B}_{\gamma'} - \mathbb{B}_\gamma)\| = \left\| \frac{1}{\epsilon_0 \epsilon_1} \begin{pmatrix} \epsilon_0 & -\epsilon_c \\ 0 & \epsilon_1 \end{pmatrix} \left[\begin{pmatrix} i\gamma'_1 & i\gamma'_c \\ 0 & i\gamma_0 \end{pmatrix} - \begin{pmatrix} i\gamma_1 & i\gamma_c \\ 0 & i\gamma_0 \end{pmatrix} \right] \right\| \quad (3.41)$$

$$\leq \left\| \frac{1}{\epsilon_0 \epsilon_1} \begin{pmatrix} \epsilon_0 & -\epsilon_c \\ 0 & \epsilon_1 \end{pmatrix} \begin{pmatrix} i(\gamma'_1 - \gamma_1) & i(\gamma'_1 - \gamma_1) \\ 0 & 0 \end{pmatrix} \right\| \quad (3.42)$$

$$= \left\| \frac{1}{\epsilon_0 \epsilon_1} \begin{pmatrix} i\epsilon_0(\gamma'_1 - \gamma_1) & i\epsilon_0(\gamma'_1 - \gamma_1) \\ 0 & 0 \end{pmatrix} \right\| \quad (3.43)$$

$$\leq \frac{2}{\inf_{[0,1]} \epsilon_1} \sup_{[0,1]} |\gamma'_1 - \gamma_1|. \quad (3.44)$$

We proved the following stability among the interior transmission eigenvalues.

Theorem 3.5 *Let n_1 be a refraction index satisfying the assumption in (1.2). Let k_j , $j = 0, \dots, l$ be any $l + 1$ interior transmission eigenvalues, counted according to its multiplicity, corresponding to (1.2) in absorbing media, $\epsilon_1 > 0$, $\gamma_1 > 0$. Then, for any $\sigma > 0$, there exists a*

$$\eta := \frac{1}{\inf_{[0,1]} \epsilon_1} \sup_{[0,1]} |\gamma'_1 - \gamma_1|, \quad (3.45)$$

depending on σ , such that there exists $l + 1$ interior transmission eigenvalues of (1.2) in absorbing media, $\epsilon_1 > 0$, $\gamma'_1 > 0$, in the σ -neighborhood of k_j , $j = 0, \dots, l$, whenever $\eta = \frac{1}{\inf_{[0,1]} \epsilon_1} \sup_{[0,1]} |\gamma'_1 - \gamma_1|$ is small enough.

This explains the perturbation theory for finitely many eigenvalues. To study the asymptotic behavior, we analyze the asymptotic behavior of zeros of $y^1(1; k)$. We need the following counting lemma.

Lemma 3.6 *If $|z - j\pi| \geq \delta > 0$, $j \in \mathbb{Z}$, then*

$$\exp\{|\Im z|\} < \frac{O(1)}{\delta} |\sin z|. \quad (3.46)$$

Proof We modify the one in [11]. Let $|z| \geq \frac{\pi}{n}$, where n is to be chosen. We discuss two cases: $0 \leq x \leq \frac{\pi}{2n}$ and $\frac{\pi}{2n} \leq x \leq \frac{\pi}{2}$.

Now $x^2 + y^2 = |z|^2$ and $0 \leq x \leq \frac{\pi}{2n}$. It implies that

$$y^2 = |z|^2 - x^2 \geq \left(\frac{\pi}{n}\right)^2 - \left(\frac{\pi}{2n}\right)^2 = \frac{3}{4} \frac{\pi^2}{n^2}. \quad (3.47)$$

Hence,

$$\cosh^2 y \geq 1 + y^2 \geq 1 + \frac{3}{4} \frac{\pi^2}{n^2} \geq [1 + \frac{1}{4} (\frac{\pi}{n})^2] \cos^2 x. \quad (3.48)$$

For $\frac{\pi}{2n} \leq x \leq \frac{\pi}{2}$, we see $\cos \frac{\pi}{2n} \geq \cos x \geq \cos \frac{\pi}{2} = 0$ and

$$\frac{1}{\cos x} = 1 + \frac{1}{2!} x^2 + (\frac{1}{4} - \frac{1}{4!}) x^4 + \dots \geq 1 + \frac{x^2}{2}, \text{ when } x \text{ is small.} \quad (3.49)$$

Hence,

$$\frac{1}{\cos^2 x} \geq 1 + x^2 + \frac{x^4}{4}, \text{ when } x \text{ is small.} \quad (3.50)$$

Using this with $\cosh^2 y \geq 1$,

$$\cosh^2 y \geq \frac{1}{\cos^2 \frac{\pi}{2n}} \cos^2 x \geq [1 + (\frac{\pi}{2n})^2 + \frac{1}{4} (\frac{\pi}{2n})^4] \cos^2 x \geq [1 + \frac{1}{4} (\frac{\pi}{n})^2] \cos^2 x. \quad (3.51)$$

To (3.48) and (3.51), we use $|\sin z|^2 = \cosh^2 y - \cos^2 x$. Hence,

$$|\sin z|^2 \geq \cosh^2 y - (1 + \frac{1}{4} \frac{\pi^2}{n^2})^{-1} \cosh^2 y \geq (1 - (1 + \frac{1}{4} \frac{\pi^2}{n^2})^{-1}) \frac{e^{2|y|}}{4}. \quad (3.52)$$

Hence, letting $C_n = 2[1 - (1 + \frac{1}{4} \frac{\pi^2}{n^2})^{-1}]^{-\frac{1}{2}} = \frac{2n\sqrt{\pi^2+4}}{\pi} = O(n)$, we conclude

$$\exp\{|\Im y|\} < C_n |\sin z|. \quad (3.53)$$

Considering $\delta = \frac{1}{n}$, this proves the lemma. \square

Proposition 3.7 *Let z_j , be the zeros of $y(1; k)$ and z'_j be the zeros of $y'(1; k)$. The following asymptotics hold.*

$$z_j = \frac{j\pi}{B} - \frac{iD}{B} + O(\frac{1}{j}), \quad j \in \mathbb{Z}; \quad (3.54)$$

$$z'_j = \frac{(j - \frac{1}{2})\pi}{B} - \frac{iD}{B} + O(\frac{1}{j}), \quad j \in \mathbb{Z}. \quad (3.55)$$

Proof We consider the zeros of $k[\epsilon_1(0)\epsilon_1(1)]^{\frac{1}{4}}y(1; k)$ instead. We observe from (1.20) that

$$|k[\epsilon_1(0)\epsilon_1(1)]^{\frac{1}{4}}y(1; k) - \sin\{kB + iD\}| = O(\frac{1}{|k|})O(\exp\{|\Im k|B\}) \quad (3.56)$$

$$= O(\frac{1}{|k|})\exp\{|\Im k|B\}, \quad \Im k \neq 0. \quad (3.57)$$

Firstly we apply the Rouché's theorem inside the strip with boundary: $\Re k = \frac{(j - \frac{1}{2})\pi}{B}$, $\Re k = \frac{(j + \frac{1}{2})\pi}{B}$. For this purpose, we choose M large in the

$$C_{j/M} = O(j/M); \quad |k| = j \quad (3.58)$$

such that (3.53) and (3.57) imply

$$|k[\epsilon_1(0)\epsilon_1(1)]^{\frac{1}{4}}y(1; k) - \sin\{kB + iD\}| < |\sin\{kB + iD\}|, \quad \Im k \neq 0. \quad (3.59)$$

The contour applies even without the behavior at $\Im k = 0$. Hence, we know $y^1(1; k)$ has a zero inside the strip.

Secondly, we apply Rouché's theorem again over a contour as the boundary of the vertical strip with a punctured hole: $\Re k = \frac{(j - \frac{1}{2})\pi}{B}$, $\Re k = \frac{(j + \frac{1}{2})\pi}{B}$ and $|k - (\frac{j\pi}{B} - i\frac{D}{B})| = \rho$. We may choose j large and some M such that

$$\rho > \frac{M}{jB} \text{ and } |C_{j/M}O(\frac{1}{j})| < 1. \quad (3.60)$$

In such a punctured strip,

$$|k[\epsilon_1(0)\epsilon_1(1)]^{\frac{1}{4}}y^1(1;k) - \sin\{kB + iD\}| < |\sin\{kB + iD\}|, \Im k \neq 0, \quad (3.61)$$

by Lemma 3.6. Hence, the zeros of $k[\epsilon_1(0)\epsilon_1(1)]^{\frac{1}{4}}y^1(1;k)$ are the same ones of the $\sin\{kB + iD\}$ inside the strip, but outside the ρ -ball centered at $\frac{j\pi}{B} - i\frac{D}{B}$, for j large. There is no zero there.

Hence, for each large j , we know there is a zero in each strip and inside the ρ -ball centered at $\frac{j\pi}{B} - i\frac{D}{B}$. That is

$$z_j = \frac{j\pi}{B} - \frac{iD}{B} + O\left(\frac{1}{j}\right), j \in \mathbb{Z}. \quad (3.62)$$

Similarly, from (1.21) we have

$$|k[\frac{\epsilon_1(0)}{\epsilon_1(1)}]^{\frac{1}{4}}y'(1;k) - \cos\{kB + iD\}| = O\left(\frac{1}{|k|}\right) \exp\{|\Im k|B\}, \Im k \neq 0. \quad (3.63)$$

We apply Lemma 3.6 as follows: $|z + \frac{\pi}{2} - j\pi| \geq \delta > 0$, $j \in \mathbb{Z}$, then

$$\exp\{|\Im z|\} < \frac{O(1)}{\delta} |\cos z|. \quad (3.64)$$

Apply Rouché's theorem again with (3.63) and (3.64), we prove the asymptotics (3.55). \square

Accordingly, we can prove the following stability result for common zeros.

Lemma 3.8 *Under the assumption of Theorem 1.1, we let $y^1(1; z)$ be the solution defined by the index $n_1^1 = \epsilon_1^1 + i\frac{\gamma_1^1}{k}$, with zeros $\{z_j\}_{j \in \mathbb{Z}}$; $(y^1)'(1; z)$ with zeros $\{z'_j\}_{j \in \mathbb{Z}}$. Then, $\{z_j\}_{j \in \mathbb{Z}}$, $\{z'_j\}_{j \in \mathbb{Z}}$ are continuous with respect to the perturbation γ_1^1 in maximal norm with ϵ_1^1 fixed.*

Proof Let $\{z_j\}_{j \in \mathbb{Z}}$ be the common zeros of $y^1(1; z)$ and $y^2(1; z)$ to index $n_1^1 := \epsilon_1^1 + i\frac{\gamma_1^1}{k}$; $\{\bar{z}_j\}_{j \in \mathbb{Z}}$ zeros of $\bar{y}^1(1; z)$ and $y^2(1; z)$ to the index $\bar{n}_1^1 := \epsilon_1^1 + i\frac{\bar{\gamma}_1^1}{k}$. Apply Proposition 3.7, we have

$$|z_j - \bar{z}_j| \leq \frac{|D^1 - \bar{D}^1|}{B^1} + |O(\frac{1}{j})|. \quad (3.65)$$

For any $\epsilon > 0$, we choose $j_0 \in \mathbb{N}$ large such that $|O(\frac{1}{j})| < \frac{\epsilon}{2}$ whenever $|j| > j_0$. Subsequently, we choose $\|\bar{\gamma}^1 - \gamma^1\|_\infty$ small such that

$$|D^1 - \bar{D}^1| \leq \frac{1}{2} \int_0^1 \frac{|\bar{\gamma}_1^1(\rho) - \gamma_1^1(\rho)|}{\sqrt{\epsilon_1^1(\rho)}} d\rho \leq C \|\bar{\gamma}^1 - \gamma^1\|_\infty < \frac{\epsilon B^1}{2}.$$

On the other hand, for zeros $\{z_{-j_0}, \dots, z_{j_0}\}$, we apply Theorem 3.5. Because that $\{z_j\}$ are zeros,

$$y^1(1; z_j) = 0, \forall |j| \leq j_0. \quad (3.66)$$

Under the result (3.10), we set $\mathcal{Y}(1; k) \equiv 0$ in system (3.15)- (3.18). From (3.15) and (3.16) respectively, we obtain for any interior transmission eigenvalue k that

$$(y^2)''(r; k) + (k^2 \epsilon_1^2 + ik\gamma_1^2)y^2(r; k) = 0, 0 \leq r \leq 1; \quad (3.67)$$

$$y^2(1; k) = 0. \quad (3.68)$$

Hence, any such k is an eigenvalue to the Dirichlet problem (3.67) and (3.68). Conversely, under the result (3.10), $\{z_j\}$ are interior transmission eigenvalues. Therefore, $\{z_j\}$ enjoy the perturbation theory provided by Theorem 3.5. Accordingly, we choose η small such that

$$|z_j - \bar{z}_j| < \epsilon, |j| < j_0. \quad (3.69)$$

This is one stability result.

To show the stability to the common zeros of $\{z'_j\}$, we repeat the same argument for zeros $\{z'_j\}$ such that

$$(y^1)'(1; z'_j) = 0, \forall j. \quad (3.70)$$

The stability result follows. \square

Finally, let the point set

$$F^m := \{z_j^m\}_{\{j \in \mathbb{Z}\}}, m \in \mathbb{N}, \quad (3.71)$$

be the collection of the common zeros of $y^l(1; k, m)$ defined by index $n^m = \epsilon_1^l + i\frac{\gamma_1^l}{k}/m$, $l = 1, 2$. We see F^1 are common zeros of $y^1(1; k)$ and $y^2(1; k)$.

Similarly, we define

$$F^{m'} := \{z_j^{m'}\}_{\{j \in \mathbb{Z}\}}, m \in \mathbb{N}, \quad (3.72)$$

as the common zeros of $(y^1)'(1; k, m)$ and $(y^2)'(1; k, m)$. Initially,

$$y^1(1; z_j) = y^2(1; z_j) = 0, z_j \in F^1; \quad (3.73)$$

$$(y^1)'(1; z_j') = (y^2)'(1; z_j') = 0, z_j' \in F^{1'}. \quad (3.74)$$

From Lemma 3.8, every elements in $F^m, F^{m'}$ moves continuously as the absorbing γ_1^1 and γ_1^2 vanishing to the zero when $m \rightarrow \infty$. Therefore, when $\gamma_1^1 \equiv \gamma_1^2 \equiv 0$,

$$y^1(1; z_j^\infty) = y^2(1; z_j^\infty) = 0, z_j^\infty \in F^\infty; \quad (3.75)$$

$$(y^1)'(1; z_j^{\infty'}) = (y^2)'(1; z_j^{\infty'}) = 0, z_j^{\infty'} \in F^{\infty'}. \quad (3.76)$$

We observe that $\{z_j^\infty\}$ are all the eigenvalues to the problem

$$\begin{cases} (y^i)'' + k^2 \epsilon_1^i(r) y^i = 0, & 0 \leq r \leq 1; \\ y^i(0) = 0, & y^i(1) = 0, \end{cases} \quad (3.77)$$

for both refraction indices; $\{z_j^{\infty'}\}$ are all the eigenvalues to the problem

$$\begin{cases} (y^i)'' + k^2 \epsilon_1^i(r) y^i = 0, & 0 \leq r \leq 1; \\ y^i(0) = 0, & (y^i)'(1) = 0, \end{cases} \quad (3.78)$$

for both refraction indices. The zero set indicator function of $F^\infty, F^{\infty'}$ defined in (3.7) are

$$H_{F^\infty}(\theta) = H_{F^{\infty'}}(\theta) = B^1 |\sin \theta|, \quad (3.79)$$

which is inferred from (3.54) with Cartwright's theory.

We have reduced to an absorbing problem to a non-absorbing one. The rest of proof is the application of the inverse rod density problem which we refer to the Corollary 2.9 in [1] provided (3.75) and (3.76). This proves the Theorem 1.1. \square

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